

# Descent Methods for Equilibrium Problems in a Banach Space

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**Abstract**—In this paper, we consider equilibrium problems with differentiable bifunctions in a Banach space setting and investigate properties of gap functions for such problems. We suggest a derivative-free descent method and give conditions which provide strong convergence of the method.  
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## 1. INTRODUCTION

Let  $U$  be a nonempty closed convex subset of a reflexive Banach space  $E$  and  $h : U \times U \rightarrow R$  an equilibrium bifunction, i.e.,  $h(u, u) = 0$  for every  $u \in U$ . Then, one can define the *equilibrium problem* (EP) that is to find an element  $\bar{u} \in U$  such that

$$h(\bar{u}, v) \geq 0, \quad \forall v \in U. \quad (1)$$

This problem was investigated by many researchers, both in finite- and infinite-dimensional spaces; e.g., see [1–4] and references therein. By introducing a gap function, one can reduce EP (1) to a scalar optimization problem; see [2,5]. At the same time, in order to find a solution to EP (1) by solving the corresponding optimization problem, it is necessary for this problem

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to coincide with its necessary optimality condition. Then, there are no local minima which are different from global ones. In the case of variational inequalities, this problem was investigated by many researchers; e.g., see [6] and references therein. In particular, it is known that it is possible to replace a strongly monotone variational inequality with a differentiable optimization problem which has no local minima and construct various feasible descent methods. In this work, we intend to present such a gap function for EP (1) and give conditions which provide convergence for the corresponding derivative-free descent method.

## 2. GAP FUNCTION FOR EQUILIBRIUM PROBLEMS

So, we consider EP (1) where, in addition,  $h(u, \cdot)$  is assumed to be convex for each  $u \in U$  and  $h$  is assumed to be differentiable. We denote by  $U^*$  the set of solutions to EP (1).

Fix  $\alpha > 0$  and let us consider an auxiliary bifunction  $\varphi : U \times U \rightarrow R$  which satisfies the following properties:

- (H<sub>0</sub>)  $\varphi$  is an equilibrium bifunction, i.e.,  $\varphi(u, u) = 0$ , for each  $u \in U$ ;
- (H<sub>1</sub>)  $\varphi(u, \cdot)$  is differentiable;
- (H<sub>2</sub>)  $\varphi(u, v) > 0$ , for all  $u \neq v$ ;
- (H<sub>3</sub>)  $\varphi(u, \cdot)$  is strongly convex with constant  $2\kappa > 0$ , for each  $u \in U$ , i.e.,

$$\varphi(u, w) - \varphi(u, v) \geq \langle \varphi'_v(u, v), w - v \rangle + \kappa \|w - v\|^2, \quad \forall w, v \in U;$$

- (H<sub>4</sub>)  $\varphi'_v(u, u) \in N(U, u) = \{q \in E^* \mid \langle q, v - u \rangle \leq 0, \forall v \in U\}$ , for each  $u \in U$ .

These conditions can be viewed as modifications and extensions of properties of auxiliary functions for variational inequalities from [7].

Now we can define the perturbed EP which is to find an element  $\bar{u} \in U$ , such that

$$h(\bar{u}, v) + \alpha\varphi(\bar{u}, v) \geq 0, \quad \forall v \in U. \quad (2)$$

We denote by  $U^\alpha$  the solution set of EP (2). Our analysis will be based on the following equivalence result between (1) and (2).

**PROPOSITION 2.1.** *Problems (1) and (2) are equivalent.*

**PROOF.** The inclusion  $U^* \subseteq U^\alpha$  follows from (H<sub>2</sub>). To prove the converse, fix  $u^* \in U^\alpha$ , then, due to the convexity of  $h(u, \cdot) + \alpha\varphi(u, \cdot)$ , we have

$$\langle h'_v(u^*, u^*) + \alpha\varphi'_v(u^*, u^*), w - u^* \rangle \geq 0, \quad \forall w \in U.$$

Using (H<sub>4</sub>) in this inequality gives

$$\langle h'_v(u^*, u^*), w - u^* \rangle \geq 0, \quad \forall w \in U,$$

which is equivalent to  $u^* \in U^*$ . ■

Since the function  $h(u, \cdot) + \alpha\varphi(u, \cdot)$  is strongly convex for each  $u \in U$ , the optimization problem

$$\min_{v \in U} \rightarrow \{h(u, v) + \alpha\varphi(u, v)\}$$

has a unique solution, which will be denoted by  $v_\alpha(u)$ . Set

$$\begin{aligned} \mu_\alpha(u) &= \sup_{v \in U} \{-h(u, v) - \alpha\varphi(u, v)\} \\ &= -h(u, v_\alpha(u)) - \alpha\varphi(u, v_\alpha(u)). \end{aligned} \quad (3)$$

We intend to show that  $\mu_\alpha$ , defined by (3), is a gap function for EP (1).

PROPOSITION 2.2.

- (i)  $\mu_\alpha(u) \geq 0$ , for all  $u \in U$ .
- (ii)  $\mu_\alpha(\bar{u}) = 0$  and  $\bar{u} \in U$  imply  $\bar{u} \in U^*$ .

PROOF. Part (i) follows from the fact that  $h + \alpha\varphi$  is an equilibrium bifunction. Next, suppose that  $\bar{u} \in U$  and  $\mu_\alpha(\bar{u}) = 0$ . Then, by definition, we have

$$-h(\bar{u}, v) - \alpha\varphi(\bar{u}, v) \leq 0, \quad \forall v \in U,$$

i.e.,  $\bar{u} \in U^\alpha = U^*$  in view of Proposition 2.1 and (ii) holds, too. ■

The result of Proposition 2.2(ii) can be strengthened.

PROPOSITION 2.3. *The following assertions are equivalent:*

- (a)  $u^*$  solves EP (1);
- (b)  $u^* = v_\alpha(u^*)$ ;
- (c)  $\mu_\alpha(u^*) = 0$  and  $u^* \in U$ .

PROOF. The implication (c)  $\Rightarrow$  (a) has been proven in Proposition 2.2(ii). Suppose that (a) holds, then, by Proposition 2.1,  $u^* \in U^\alpha$  and

$$\langle h'_v(u^*, u^*) + \alpha\varphi'_v(u^*, u^*), v^* - u^* \rangle \geq 0,$$

where  $v^* = v_\alpha(u^*)$ . At the same time, we have

$$\langle h'_v(u^*, v^*) + \alpha\varphi'_v(u^*, v^*), u^* - v^* \rangle \geq 0.$$

Adding both inequalities gives

$$\alpha \langle \varphi'_v(u^*, u^*) - \varphi'_v(u^*, v^*), v^* - u^* \rangle \geq \langle h'_v(u^*, v^*) - h'_v(u^*, u^*), v^* - u^* \rangle \geq 0,$$

since  $h'_v(u^*, \cdot)$  is monotone. But  $\varphi'_v(u^*, \cdot)$  is strongly monotone with constant  $\kappa$  and we get

$$-\alpha\kappa \|v^* - u^*\|^2 \geq 0,$$

i.e.,  $u^* = v_\alpha(u^*)$ , hence (a)  $\Rightarrow$  (b). Next, suppose that  $u^* = v_\alpha(u^*)$ , then, by definition,

$$\mu_\alpha(u^*) = -h(u^*, u^*) - \alpha\varphi(u^*, u^*) = 0,$$

and hence, (b)  $\Rightarrow$  (c) and the proof is complete. ■

From the assertions of Propositions 2.2 and 2.3 it follows that the initial EP (1) is equivalent to the following optimization problem:

$$\min_{u \in U} \rightarrow \mu_\alpha(u), \tag{4}$$

which, however, can have local minima. Therefore, we aim to give conditions under which EP (1) is equivalent to the following problem:

$$\mu'_\alpha(\bar{u}, v - \bar{u}) \geq 0, \quad \forall v \in U. \tag{5}$$

Clearly, (5) is a necessary optimality condition for (4). If (4) is equivalent to (5) when it is possible to construct a descent method with respect to  $\mu_\alpha$  which will converge to a solution to EP (1) under a suitable choice of the stepsize. However, such an equivalence result needs additional assumptions, which will be considered in the next section.

3. OPTIMALITY CONDITIONS AND DESCENT PROPERTIES

First we give several additional assumptions on  $h$ .

(A<sub>1</sub>) For each pair of points  $u', u'' \in U$ , we have

$$\langle h'_v(u', u') - h'_v(u'', u''), u' - u'' \rangle \geq \tau' \|u' - u''\|^2$$

for some  $\tau' > 0$ .

(A<sub>2</sub>) For each pair of points  $u, v \in U$ , we have

$$\langle h'_u(u, v) + h'_v(u, v), v - u \rangle \geq \tau'' \|v - u\|^2$$

for some  $\tau'' > 0$ .

(A<sub>3</sub>)  $h'_u(u, v)$  and  $h'_v(u, v)$  are uniformly Lipschitz continuous on  $U \times U$ .

Assumptions (A<sub>1</sub>) and (A<sub>2</sub>) can be viewed as some variants of strong monotonicity for bifunctions. Recall that a bifunction  $f : U \times U \rightarrow R$  is said to be

(a) *monotone* if for each pair of points  $u, v \in U$ , we have

$$f(u, v) + f(v, u) \leq 0;$$

(b) *strongly monotone* with constant  $\beta > 0$ , if for each pair of points  $u, v \in U$ , we have

$$f(u, v) + f(v, u) \leq -\beta \|u - v\|^2$$

(see, e.g., [8, Definition 2.1.5]).

It is known that convexity of  $h(u, \cdot)$  and strong monotonicity of  $h$  implies (A<sub>1</sub>), i.e.,  $h'_v(u, u)$  is then strongly monotone, see [8, Proposition 2.1.17]. Let us consider the case where

$$h(u, v) = \langle G(u), v - u \rangle,$$

$G : U \rightarrow E^*$  is a differentiable strongly monotone mapping. Then (1) becomes a variational inequality problem and (A<sub>1</sub>), (A<sub>2</sub>) hold. In fact, we have

$$h'_v(u, v) = G(u), \quad h'_u(u, v) = \nabla G(u)^\top (v - u) - G(u),$$

i.e., (A<sub>1</sub>) is fulfilled. Next,

$$\langle h'_u(u, v) + h'_v(u, v), v - u \rangle = \langle \nabla G(u)^\top (v - u), v - u \rangle \geq \beta' \|v - u\|^2,$$

for some  $\beta' > 0$ , i.e., (A<sub>2</sub>) is also fulfilled. Nevertheless, (A<sub>1</sub>) and (A<sub>2</sub>) are not equivalent in general, as the following simple examples illustrate.

EXAMPLE 1. Let  $E = R$ ,  $h(u, v) = u^{-2}(v^4 - u^4)$ . Then  $h(u, \cdot)$  is convex and

$$[h'_v(p, p) - h'_v(q, q)](p - q) = 4(p - q)^2,$$

i.e., (A<sub>1</sub>) holds. At the same time, we have

$$\begin{aligned} [h'_v(u, v) + h'_u(u, v)](v - u) &= [4u^{-2}v^3 - 2u^{-1}v^4 - 2u](v - u) \\ &= u^{-2}(4v^3 - 2uv^4 - 2u^3)(v - u) < 0 \end{aligned}$$

if, for example, we set  $u = 1, v = 10$ . Hence, (A<sub>2</sub>) does not hold.

EXAMPLE 2. Let  $E = R$ ,  $h(u, v) = u^2(v^4 - u^4)$ . Then  $h(u, \cdot)$  is convex,  $h$  is monotone, since

$$\begin{aligned} h(u, v) + h(v, u) &= u^2(v^4 - u^4) + v^2(u^4 - v^4) = (u^2 - v^2)(v^4 - u^4) \\ &= -(u^2 - v^2)^2(u^2 + v^2) < 0. \end{aligned}$$

For this reason,  $h'_v(u, u)$  is monotone (see [8, Proposition 2.1.17]). At the same time, we have  $[h'_v(u, v) + h'_u(u, v)](v - u) = (4u^2v^3 + 2uv^4 - 6u^5)(v - u) = 2u^2(2v^3 + v^4u^{-1} - 3u^3)(v - u) < 0$  if we set  $u = -1, v = 10$ . Hence, (A<sub>2</sub>) does not hold.

Next, we suppose that the auxiliary bifunction  $\varphi$  satisfies the following additional conditions.

(H<sub>5</sub>)  $\langle \varphi'_v(u, v) + \varphi'_u(u, v), v - u \rangle \geq 0$ , for all  $u, v \in U$ .

(H<sub>6</sub>)  $\varphi'_v(u, v)$  and  $\varphi'_u(u, v)$  are uniformly Lipschitz continuous on  $U \times U$ .

Clearly, (H<sub>5</sub>) is a weakened version of (A<sub>1</sub>) with respect to  $\varphi$ .

**THEOREM 3.1.** *Suppose that  $(A_2)$  is fulfilled. Then:*

- (i) *if (5) holds for some  $\bar{u} \in U$ , then  $\bar{u} \in U^*$ ;*
- (ii)

$$\mu'_\alpha(u, v_\alpha(u) - u) \leq -\tau'' \|v_\alpha(u) - u\|^2, \quad (6)$$

for each  $u \in U$ .

**PROOF.** By definition,

$$\mu'_\alpha(u, d) = -\langle h'_u(u, v_\alpha(u)), d \rangle - \alpha \langle \varphi'_u(u, v_\alpha(u)), d \rangle.$$

Set  $d = v_\alpha(u) - u$ , then

$$\langle h'_v(u, v_\alpha(u)) + \alpha \varphi'_v(u, v_\alpha(u)), d \rangle \leq 0$$

since  $v_\alpha(u)$  solves (3), moreover, by  $(H_5)$ ,

$$\langle \varphi'_v(u, v_\alpha(u)) + \varphi'_u(u, v_\alpha(u)), d \rangle \leq 0.$$

Therefore,

$$\begin{aligned} \mu'_\alpha(u, d) &= -\langle h'_v(u, v_\alpha(u)) + h'_u(u, v_\alpha(u)), d \rangle \\ &\quad + \langle h'_v(u, v_\alpha(u)) + \alpha \varphi'_v(u, v_\alpha(u)), d \rangle \\ &\quad - \alpha \langle \varphi'_v(u, v_\alpha(u)) + \varphi'_u(u, v_\alpha(u)), d \rangle \\ &\leq -\langle h'_v(u, v_\alpha(u)) + h'_u(u, v_\alpha(u)), d \rangle. \end{aligned}$$

Now, applying  $(A_2)$  in this inequality gives (6). Next, combining (5) and (6) with  $u = \bar{u}$ , we have

$$0 \leq \mu'_\alpha(\bar{u}, v_\alpha(\bar{u}) - \bar{u}) < 0$$

if  $v_\alpha(\bar{u}) \neq \bar{u}$ , a contradiction. Hence, (5) now implies  $\bar{u} = v_\alpha(\bar{u})$ , i.e.,  $\bar{u} \in U^*$  due to Proposition 2.3. The proof is complete.  $\blacksquare$

Now we show that  $\mu_\alpha(u)$  is a majorant for the distance to a solution.

**PROPOSITION 3.1.** *If  $(A_1)$  holds, then EP (1) has a unique solution.*

**PROOF.** Clearly, (1) is equivalent to the variational inequality

$$\langle G(u^*), v - u^* \rangle \geq 0, \quad \forall v \in U,$$

with  $G(u) = h'_v(u, u)$ , which has a unique solution since  $G$  is now strongly monotone; see, e.g., [8, Proposition 1.1.4, Theorem 2.1.2].  $\blacksquare$

**THEOREM 3.2.** *Suppose that  $(A_1)$  and  $(A_3)$  hold. Then, there exists a constant  $\sigma > 0$ , such that*

$$\mu_\alpha(u) \geq \sigma \|u - \bar{u}\|^2, \quad \forall u \in U, \quad (7)$$

where  $\bar{u}$  is a unique solution to EP (1).

**PROOF.** From Proposition 3.1 it follows that EP (1) has the unique solution  $\bar{u}$  and we have

$$\langle h'_v(\bar{u}, \bar{u}), v_\alpha(u) - \bar{u} \rangle \geq 0$$

and

$$\langle h'_v(u, v_\alpha(u)) + \alpha \varphi'_v(u, v_\alpha(u)), \bar{u} - v_\alpha(u) \rangle \geq 0$$

for each  $u \in U$ . Adding these inequalities gives

$$\langle h'_v(\bar{u}, \bar{u}) - h'_v(u, v_\alpha(u)), v_\alpha(u) - \bar{u} \rangle + \alpha \langle \varphi'_v(u, v_\alpha(u)), \bar{u} - v_\alpha(u) \rangle \geq 0.$$

Taking into account  $(A_1)$  and  $(H_4)$ , we now have

$$\begin{aligned} \tau' \|u - \bar{u}\|^2 &\leq \langle h'_v(u, u) - h'_v(\bar{u}, \bar{u}), u - \bar{u} \rangle \\ &\leq \langle h'_v(u, u) - h'_v(u, v_\alpha(u)), u - \bar{u} \rangle \\ &\quad - \langle h'_v(u, v_\alpha(u)) - h'_v(\bar{u}, \bar{u}), v_\alpha(u) - u \rangle \\ &\quad + \alpha \langle \varphi'_v(u, v_\alpha(u)) - \varphi'_v(u, u), \bar{u} - u \rangle + \alpha \langle \varphi'_v(u, v_\alpha(u)), u - v_\alpha(u) \rangle. \end{aligned}$$

In addition,

$$\begin{aligned} \langle \varphi'_v(u, v_\alpha(u)), u - v_\alpha(u) \rangle &\leq \varphi(u, u) - \varphi(u, v_\alpha(u)) - \kappa \|u - v_\alpha(u)\|^2 \\ &\leq 0 \end{aligned}$$

because of  $(H_0)$ ,  $(H_2)$ , and  $(H_3)$ . Hence,

$$\begin{aligned} \tau' \|u - \bar{u}\|^2 &\leq L_h \|u - v_\alpha(u)\| \|u - \bar{u}\| + L_h \|u - \bar{u}\| \|v_\alpha(u) - u\| \\ &\quad + L_h \|v_\alpha(u) - \bar{u}\| \|v_\alpha(u) - u\| + \alpha L_\varphi \|v_\alpha(u) - u\| \|u - \bar{u}\| \\ &\leq (3L_h + \alpha L_\varphi) \|u - v_\alpha(u)\| \|u - \bar{u}\| + L_h \|v_\alpha(u) - u\|^2, \end{aligned}$$

where  $L_h$  and  $L_\varphi$  are the corresponding Lipschitz constants for  $h'_v(w, \cdot)$  and  $\varphi'_v(w, \cdot)$ , respectively. It follows that there exists a constant  $C' > 0$ , such that

$$\|u - \bar{u}\| \leq C' \|u - v_\alpha(u)\|.$$

On the other hand, since the function

$$\Phi_\alpha(u, \cdot) = h(u, \cdot) + \alpha \varphi(u, \cdot)$$

is strongly convex with constant  $2\alpha\kappa$ , we have

$$\mu_\alpha(u) = \Phi_\alpha(u, u) - \Phi_\alpha(u, v_\alpha(u)) \geq \alpha\kappa \|u - v_\alpha(u)\|^2;$$

see, e.g., [9, p. 42]. Combining both the inequalities gives (7) with  $\sigma = \alpha\kappa/C'$ , as desired. ■

**COROLLARY 3.1.** *Suppose that  $(A_1)$  and  $(A_3)$  hold. Then the function  $\mu_\alpha$  has bounded level sets on  $U$ .*

We now establish the Lipschitz continuity of the mapping  $v_\alpha$  and the gradient map  $\mu'_\alpha$ .

**LEMMA 3.1.** *If  $(A_3)$  is fulfilled, then the mapping  $v_\alpha$  is Lipschitz continuous.*

**PROOF.** Fix  $u', u'' \in U$  and set  $v' = v_\alpha(u')$ ,  $v'' = v_\alpha(u'')$ . Then we have

$$\langle h'_v(u', v') + \alpha \varphi'_v(u', v'), v'' - v' \rangle \geq 0$$

and

$$\langle h'_v(u'', v'') + \alpha \varphi'_v(u'', v''), v' - v'' \rangle \geq 0.$$

Adding these inequalities gives

$$\langle h'_v(u', v') - h'_v(u'', v''), v'' - v' \rangle \geq \alpha \langle \varphi'_v(u', v') - \varphi'_v(u'', v''), v' - v'' \rangle.$$

Since  $h(u, \cdot)$  is convex and  $\varphi(u, \cdot)$  is strongly convex, it follows that

$$\begin{aligned} &\langle h'_v(u', v') - h'_v(u'', v''), v'' - v' \rangle + \alpha \langle \varphi'_v(u', v') - \varphi'_v(u'', v''), v'' - v' \rangle \\ &\geq \langle h'_v(u'', v') - h'_v(u'', v''), v' - v'' \rangle + \alpha \langle \varphi'_v(u'', v') - \varphi'_v(u'', v''), v' - v'' \rangle \\ &\geq 2\alpha\kappa \|v' - v''\|^2. \end{aligned}$$

Therefore, we have

$$(L_h + \alpha L_\varphi) \|u' - u''\| \|v' - v''\| \geq 2\alpha\kappa \|v' - v''\|^2,$$

or equivalently,

$$(L_h + \alpha L_\varphi) \|u' - u''\| \geq 2\alpha\kappa \|v' - v''\|,$$

where  $L_h$  and  $L_\varphi$  are the corresponding Lipschitz constants for  $h'_v(\cdot, w)$  and  $\varphi'_v(\cdot, w)$ , respectively. Thus,  $v_\alpha$  is Lipschitz continuous. ■

PROPOSITION 3.2. Suppose  $(A_3)$  is fulfilled. Then  $\mu'_\alpha$  is Lipschitz continuous.

PROOF. Fix  $u', u'' \in U$  and set  $v' = v_\alpha(u')$ ,  $v'' = v_\alpha(u'')$ . Then,

$$\begin{aligned} \|\mu'_\alpha(u') - \mu'_\alpha(u'')\| &= \| [h'_u(u', v') - h'_u(u'', v'')] + \alpha [\varphi'_u(u', v') - \varphi'_u(u'', v'')] \| \\ &\leq \|h'_u(u', v') - h'_u(u'', v'')\| + \|h'_u(u', v'') - h'_u(u'', v'')\| \\ &\quad + \alpha \|\varphi'_u(u', v') - \varphi'_u(u', v'')\| + \alpha \|\varphi'_u(u', v'') - \varphi'_u(u'', v'')\| \\ &\leq L_h \|v' - v''\| + L_h \|u' - u''\| + \alpha L_\varphi \|v' - v''\| + \alpha L_\varphi \|u' - u''\| \\ &\leq (L_h \cdot L_v + \alpha L_\varphi \cdot L_v + \alpha L_\varphi) \|u' - u''\|, \end{aligned}$$

where  $L_h$ ,  $L_\varphi$ , and  $L_v$  are the corresponding Lipschitz constants for  $h'_u(w, \cdot)$  and  $h'_u(\cdot, w)$ ,  $\varphi'_u(w, \cdot)$ , and  $\varphi'_u(\cdot, w)$ , and  $v_\alpha$ , which exists because of Lemma 3.1. Therefore,  $\mu'_\alpha$  is Lipschitz continuous, as desired.  $\blacksquare$

#### 4. DESCENT METHOD AND ITS CONVERGENCE

The method for solving EP (1) under the above assumptions can be described as follows.

METHOD.

STEP 0. Choose a point  $u^0 \in U$ , numbers  $\alpha > 0$ ,  $\theta \in (0, 1)$ , and  $\beta \in (0, 1)$ . Set  $k = 0$ .

STEP 1. Find  $v_\alpha(u^k)$  and set  $d^k = v_\alpha(u^k) - u^k$ .

STEP 2. Determine  $m$  as the smallest nonnegative integer such that

$$\mu_\alpha(u^k + \beta^m d^k) \leq \mu_\alpha(u^k) - \theta \beta^m \|d^k\|^2. \quad (8)$$

STEP 3. Set  $t_k = \beta^m$ ,  $u^{k+1} = u^k + t_k d^k$ ,  $k = k + 1$  and go to Step 1.

First we recall the well-known property of Lipschitz continuous gradient maps.

LEMMA 4.1. (See [10, Lemma 1.2].) Suppose that the gradient  $f'$  of a function  $f : E \rightarrow R$  is Lipschitz continuous with constant  $L$  on the set  $U$ . Then, for all  $u', u'' \in U$  and for each  $\lambda \in R$ , we have

$$f(u' + \lambda(u'' - u')) \leq f(u') + \lambda \langle f'(u'), u'' - u' \rangle + L \lambda^2 \frac{\|u'' - u'\|^2}{2}. \quad (9)$$

We are now ready to establish the convergence result for our descent method with a Armijo-type linesearch procedure. This method can be viewed as an extension of the corresponding descent methods for strongly monotone variational inequalities; e.g., see [6] and references therein.

THEOREM 4.1. Suppose that  $(A_1)$ – $(A_3)$  are fulfilled, a sequence  $\{u^k\}$  is generated by the method with  $\theta < \tau''$ . Then  $\{u^k\}$  converges strongly to a unique solution of EP (1).

PROOF. First we note that EP (1) has a unique solution under the above assumptions because of Proposition 3.1. Next, taking into account Proposition 3.2 and using (9) with  $f = \mu_\alpha$ ,  $u' = u^k$ ,  $u'' = v_\alpha(u^k)$ , we have

$$\mu_\alpha(u^k + \lambda d^k) - \mu_\alpha(u^k) \leq \lambda \langle \mu'_\alpha(u^k), d^k \rangle + L_\mu \lambda^2 \frac{\|d^k\|^2}{2},$$

where  $L_\mu$  is the Lipschitz constant for  $\mu'_\alpha$ . From (6) it now follows that

$$\begin{aligned} \mu_\alpha(u^k + \lambda d^k) - \mu_\alpha(u^k) &\leq -\tau'' \lambda \|d^k\|^2 + L_\mu \lambda^2 \frac{\|d^k\|^2}{2} \\ &= -\lambda \left( \tau'' - \frac{L_\mu \lambda}{2} \right) \|d^k\|^2 \\ &\leq -\theta \lambda \|d^k\|^2 \end{aligned}$$

if  $\tau'' - L_\mu \lambda / 2 \geq \theta$ , or equivalently,  $\lambda \leq 2(\tau'' - \theta) / L_\mu$ . Therefore, the linesearch procedure in (8) is always finite and we have

$$t_k \geq t' = \min \left\{ \beta, 2\beta \frac{(\tau'' - \theta)}{L_\mu} \right\} > 0.$$

The sequence  $\{\mu_\alpha(u^k)\}$  is decreasing and from (8) we obtain

$$\mu_\alpha(u^k) - \mu_\alpha(u^{k+1}) \geq \theta t^k \|d^k\|^2 \geq \theta t' \|d^k\|^2,$$

and hence,

$$\lim_{k \rightarrow \infty} \|u^k - v_\alpha(u^k)\| = 0.$$

Next, due to Corollary 3.1, sequence  $\{u^k\}$  is bounded, hence, so are  $\{v_\alpha(u^k)\}$ ,  $\{h'_v(u^k, v_\alpha(u^k))\}$ , and  $\{\varphi'_v(u^k, v_\alpha(u^k))\}$ . Using Proposition 2.2(i), we now obtain

$$\begin{aligned} 0 \leq \mu_\alpha(u^k) &= -h(u^k, v_\alpha(u^k)) - \alpha \varphi(u^k, v_\alpha(u^k)) \\ &= [h(u^k, u^k) - h(u^k, v_\alpha(u^k))] + \alpha [\varphi(u^k, u^k) - \varphi(u^k, v_\alpha(u^k))] \\ &\leq \tilde{L}_h \|u^k - v_\alpha(u^k)\| + \alpha \tilde{L}_\varphi \|u^k - v_\alpha(u^k)\|, \end{aligned}$$

where  $\tilde{L}_h$  and  $\tilde{L}_\varphi$  are the Lipschitz constants for  $h(u^k, \cdot)$  and  $\varphi(u^k, \cdot)$ , which are uniformly bounded. Therefore,

$$\lim_{k \rightarrow \infty} \mu_\alpha(u^k) = 0.$$

From Theorem 3.2 it follows that the sequence  $\{u^k\}$  converges strongly to the solution  $u^*$  of EP (1). The proof is complete.  $\blacksquare$

Thus, convergence properties of our method depend strongly on the differentiability of  $h$ , but the implementation of the method does not involve computations of derivatives. Therefore, our method has certain advantages over the approach which consists in preliminary transformation of EP (1) into an equivalent variational inequality and applying the known descent methods to the transformed problem.

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